Notes on exact multi-soliton solutions of noncommutative integrable hierarchies

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ABSTRACT: We study exact multi-soliton solutions of integrable hierarchies on noncommutative space-times which are represented in terms of quasi-determinants of Wronski matrices by Etingof, Gelfand and Retakh. We analyze the asymptotic behavior of the multi-soliton solutions and found that the asymptotic configurations in soliton scattering process can be all the same as commutative ones, that is, the configuration of N-soliton solution has N isolated localized energy densities and the each solitary wave-packet preserves its shape and velocity in the scattering process. The phase shifts are also the same as commutative ones. Furthermore noncommutative toroidal Gelfand-Dickey hierarchy is introduced and the exact multi-soliton solutions are given.

KEYWORDS: Integrable Hierarchies, Integrable Field Theories, Non-Commutative Geometry.

Contents

1. Introduction

Extension of integrable systems and soliton theories to non-commutative (NC) space-times[1] have been studied by many authors for the last couple of years and various kind of integrable-like properties have been revealed [1, 2]. This is partially motivated by recent developments of NC gauge theories on D-branes. In the NC gauge theories, NC extension corresponds to introduction of background magnetic fields and NC solitons are, in some situations, just lower-dimensional D-branes themselves. Hence exact analysis of NC solitons just leads to that of D-branes and various applications to D-brane dynamics have been successful [3]. In this sense, NC solitons plays important roles in NC gauge theories.

Most of NC integrable equations such as NC KdV equations apparently belong not to gauge theories but to scalar theories. However now, it is proved that they can be derived from NC anti-self-dual (ASD) Yang-Mills equations by reduction [4], which is first conjectured explicitly by the author and K. Toda [5]. (Original commutative one is proposed by R. Ward [6] and hence this conjecture is sometimes called *NC Ward's conjecture.*) Therefore analysis of exact soliton solutions of NC integrable equations could be applied to the corresponding physical situations in the framework of N=2 string theory [7 – 9].

Furthermore, some soliton equations describe real phenomena such as shallow water waves in fluid dynamics, optics and so on. If noncommutativity in space-time affects soliton dynamics, then we can check whether our universe is noncommutative or not by comparing experimental results and estimate the strength or the upper bound of the noncommutativity

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[1]In the present paper, the word "NC" always refers to generalization to noncommutative spaces, not to non-abelian and so on.

Hence, construction and analysis of exact multi-soliton solutions are worth studying from various viewpoints of integrable systems, string theory, and perhaps detection of noncommutativity in our universe.

Exact multi-soliton solutions of noncommutative KP hierarchy are constructed by Etingof, Gelfand and Retakh in 1997 [10], where quasi-determinants play crucial roles. (For other applications of quasi-determinants to noncommutative integrable systems, see e.g. [11 – 14].) However, their discussion is general and explicit analysis of the behavior of their soliton solutions has not yet been done. Paniak also constructs multi-soliton solutions of NC KP and KdV equations (not hierarchies) and studies the scattering process [15].[2] However, the discussion about the soliton dynamics is mainly focused on two-soliton scatterings.

In this paper, we study exact multi-soliton solutions of NC integrable hierarchies in terms of quasi-determinants of Wronski matrices, which is developed by Etingof, Gelfand and Retakh. We analyze the asymptotic behavior of the multi-soliton solutions and found that the asymptotic configurations can be real-valued though NC fields take complex values in general. The behavior in soliton scatterings is all the same as commutative ones, that is, the N-soliton solution has N isolated localized energy densities and the each wave-packet preserve its shape and velocity in the scattering process. The phase shift is also the same as commutative one.

This paper is organized as follows. In section 2, we make a brief introduction to NC field theory in star-product formalism. In section 3 and 4, we review definition and some properties of quasi-determinants and their applications to construction of multi-soliton solutions of NC integrable hierarchy in star-product formalism. In the end of section 4, we introduce NC toroidal Gelfand-Dickey (GD) hierarchy and give exact multi-soliton solutions which are new. In section 5, we discuss the asymptotic behavior of them in detail. Section 6 is devoted to conclusion and discussion.

2. NC field theory in the star-product formalism

NC spaces are defined by the noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}, \tag{2.1}$$

where the constant θ^{ij} is called the *NC parameter*. If the coordinates are real, NC parameters should be real. Because the rank of the NC parameter is even, dimension of NC space-times must be more than two. Hence in this paper, we deal not with integrable systems in $(0 + 1)$-dimension such as the Painlevé equation, but with ones in $(1 + 1)$ or $(2 + 1)$-dimension such as the KdV and KP equations. In $(1 + 1)$-dimension, we can take only space-time noncommutativity as $[t, x] = i\theta$. In $(2+1)$-dimension, there are essentially two kind of choices of noncommutativity, that is, space-space noncommutativity: $[x, y] = i\theta$ and space-time noncommutativity: $[t, x] = i\theta$ or $[t, y] = i\theta$, where the coordinates (x, y) and t correspond to space and time coordinates, respectively.

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[2]Dimakis and Müller-Hoissen present perturbative corrections with respect to a noncommutative parameter in 2-soliton scatterings of the NC KdV equation [16] before the Paniak's work.

NC field theories are given by the replacement of ordinary products in the commutative field theories with the *star-products* and realized as deformed theories from the commutative ones. The star-product is defined for ordinary fields on flat spaces, explicitly by

$$f \star g(x) := \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x}$$
$$= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \mathcal{O}(\theta^2), \tag{2.2}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. This explicit representation is known as the *Moyal product* [17]. The ordering of fields in nonlinear terms are determined so that some structures such as gauge symmetries and Lax representations should be preserved.

The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and reduces to the ordinary product in the commutative limit: $\theta^{ij} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative," that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in general. Hence the integrability of the equations are not so trivial as commutative cases, especially for space-time noncommutativity.

3. Brief review of quasi-determinants

In this section, we make a brief introduction of quasi-determinants introduced by Gelfand and Retakh [18, 19] and present a few properties of them which play important roles in the following sections. The detailed discussion is seen in e.g. [20, 21]. Relation between quasi-determinants and NC symmetric functions is seen in e.g. [22].

Quasi-determinants are not just a generalization of usual commutative determinants but rather related to inverse matrices. From now on, we suppose existence of all the inverses.

Let $A = (a_{ij})$ be a $N \times N$ matrix and $B = (b_{ij})$ be the inverse matrix of A, that is, $A \star B = B \star A = 1$. Here all products of matrix elements are supposed to be star-products, though the present discussion hold for more general situation where the matrix elements belong to a noncommutative ring.

Quasi-determinants of A are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}. \tag{3.1}$$

In the commutative limit, this is reduced to

$$|A|_{ij} \longrightarrow (-1)^{i+j}\frac{\det A}{\det \tilde{A}^{ij}}, \tag{3.2}$$

where $\tilde{A}^{ij}$ is the matrix obtained from A deleting the i-th row and the j-th column.

We can write down more explicit form of quasi-determinants. In order to see it, let us recall the following formula for a block-decomposed square matrix:

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - B \star D^{-1} \star C)^{-1} & -A^{-1} \star B \star (D - C \star A^{-1} \star B)^{-1} \\ -(D - C \star A^{-1} \star B)^{-1} \star C \star A^{-1} & (D - C \star A^{-1} \star B)^{-1} \end{pmatrix},
$$

where A and D are square matrices. We note that any matrix can be decomposed as a 2×2 matrix by block decomposition where one of the diagonal parts is 1×1. Then the above formula can be applied to the decomposed 2×2 matrix and an element of the inverse matrix is obtained. Hence quasi-determinants can be also given iteratively by:

$$
\begin{aligned}
|A|_{ij} &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star ((\tilde{A}^{ij})^{-1})_{i'j'} \star a_{j'j} \\
&= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} \star (|\tilde{A}^{ij}|_{j'i'})^{-1} \star a_{j'j}.
\end{aligned} \tag{3.3}
$$

It is sometimes convenient to represent the quasi-determinant as follows:

$$
|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \tag{3.4}
$$

Examples of quasi-determinants are, for a 1×1 matrix $A = a$

$$
|A| = a,
$$

and for a 2×2 matrix $A = (a_{ij})$

$$
|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} \star a_{22}^{-1} \star a_{21}, \quad |A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} \star a_{21}^{-1} \star a_{22},
$$

$$
|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22} \star a_{12}^{-1} \star a_{11}, \quad |A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21} \star a_{11}^{-1} \star a_{12},
$$

and for a 3×3 matrix $A = (a_{ij})$

$$
|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \star \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \star \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}
$$

$$
= a_{11} - a_{12} \star \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{12} \star \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \star a_{31}
$$

$$
- a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \star a_{21} - a_{13} \star \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \star a_{31},
$$

and so on.

4. Exact soliton solutions of NC integrable hierarchies

In this section, we give exact multi-soliton solutions of several NC integrable hierarchies in terms of quasi-determinants. In the commutative case, determinants of Wronski matrices play crucial roles. In the NC case, these determinants are just replaced with the quasi-determinants. We review foundation of the NC KP hierarchy and the l-reduced hierarchies (so called NC GD hierarchies or NC lKdV hierarchies), and present the exact multi-soliton solutions of them developed by Etingof, Gelfand and Retakh [10]. Finally we extend their discussion to the NC toroidal GD hierarchy.

An N-th order pseudo-differential operator A is represented as follows

$$A = a_N \partial_x^N + a_{N-1} \partial_x^{N-1} + \cdots + a_0 + a_{-1} \partial_x^{-1} + a_{-2} \partial_x^{-2} + \cdots, \qquad (4.1)$$

where a_i is a function of x associated with noncommutative associative products (here, the Moyal products). When the coefficient of the highest order a_N equals to 1, we call it *monic*. Here we introduce useful symbols:

$$A_{\geq r} := \partial_x^N + a_{N-1} \partial_x^{N-1} + \cdots + a_r \partial_x^r, \qquad (4.2)$$

$$A_{\leq r} := A - A_{\geq r+1} = a_r \partial_x^r + a_{r-1} \partial_x^{r-1} + \cdots. \qquad (4.3)$$

The action of a differential operator ∂_x^n on a multiplicity operator f is formally defined as the following generalized Leibniz rule:

$$\partial_x^n \cdot f := \sum_{i \geq 0} \binom{n}{i} (\partial_x^i f) \partial^{n-i}, \qquad (4.4)$$

where the binomial coefficient is given by

$$\binom{n}{i} := \frac{n(n-1) \cdots (n-i+1)}{i(i-1) \cdots 1}. \qquad (4.5)$$

We note that the definition of the binomial coefficient (4.5) is applicable to the case for negative n, which just define the action of negative power of differential operators.

The composition of pseudo-differential operators is also well-defined and the total set of pseudo-differential operators forms an operator algebra. For a monic pseudo-differential operator A, there exist the unique inverse A^{-1} and the unique m-th root $A^{1/m}$ which commute with A. (These proofs are all the same as commutative ones.) For more on pseudo-differential operators and Sato's theory, see e.g. [23 – 26].

In order to define the NC KP hierarchy, let us introduce a Lax operator:

$$L = \partial_x + u_2 \partial_x^{-1} + u_3 \partial_x^{-2} + u_4 \partial_x^{-3} + \cdots, \qquad (4.6)$$

where the coefficients u_k ($k = 2, 3, \ldots$) are functions of infinite coordinates $\vec{x} := (x_1, x_2, \ldots)$ with $x_1 \equiv x$:

$$u_k = u_k(x_1, x_2, \ldots). \qquad (4.7)$$

The noncommutativity is introduced into the coordinates $(x_1, x_2, \ldots)$ as eq. (2.1) here.

The NC KP hierarchy is defined in Sato's framework as

$$\partial_m L = [B_m, L]_\star, \quad m = 1, 2, \ldots, \tag{4.8}$$

where the action of $\partial_m := \partial / \partial x_m$ on the pseudo-differential operator L should be interpreted to be coefficient-wise, that is, $\partial_m L := [\partial_m, L]_\star$ or $\partial_m \partial_x^k = 0$. The differential operator B_m is given by

$$B_m := (\underbrace{L \star \cdots \star L}_{m \text{ times}})_{\geq 0} =: (L^m)_{\geq 0}. \tag{4.9}$$

The KP hierarchy gives rise to a set of infinite differential equations with respect to infinite kind of fields from the coefficients in eq. (4.8) for a fixed m. Hence it contains huge amount of differential (evolution) equations for all m. The l.h.s. of eq. (4.8) becomes $\partial_m u_k$ which shows a kind of flow in the x_m direction. In the x_2-flow equations, we can see that infinite kind of fields $u_3, u_4, u_5, \ldots$ are represented in terms of one kind of field u_2 [27, 28].

If we put the constraint $(L^l)_{\leq -1} = 0$ or equivalently $L^l = B_l$ on the NC KP hierarchy (4.8), we get a reduced NC KP hierarchy which is called the *l-reduction* of the NC KP hierarchy, or the *NC lKdV hierarchy*, or the l-th *NC Gelfand-Dickey hierarchy*. Especially, the 2-reduction of the NC KP hierarchy is just the NC KdV hierarchy. Explicit examples are seen in e.g. [28]. (See also [29 – 31].)

Now we construct multi-soliton solutions of the NC KP hierarchy. Let us introduce the following functions,

$$f_s(\vec{x}) = e_\star^{\xi(\vec{x}; \alpha_s)} + a_s e_\star^{\xi(\vec{x}; \beta_s)}, \tag{4.10}$$

where

$$\xi(\vec{x}; \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots, \tag{4.11}$$

and α_s, β_s and a_s are constants. Star exponential functions are defined by

$$e_\star^{f(x)} := 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{f(x) \star \cdots \star f(x)}_{n \text{ times}}. \tag{4.12}$$

An N-soliton solution of the NC KP hierarchy (4.8) is given by [10],

$$L = \Phi_N \star \partial_x \Phi_N^{-1}, \tag{4.13}$$

where

$$\Phi_N \star f = |W(f_1, \ldots, f_N, f)|_{N+1, N+1},$$

$$= \begin{vmatrix} f_1 & f_2 & \cdots & f_N & f \\ f_1' & f_2' & \cdots & f_N' & f' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} & f^{(N-1)} \\ f_1^{(N)} & f_2^{(N)} & \cdots & f_N^{(N)} & \boxed{f^{(N)}} \end{vmatrix}. \tag{4.14}$$

The Wronski matrix $W(f_1, f_2, \cdots, f_m)$ is given by

$$W(f_1, f_2, \cdots, f_m) := \begin{pmatrix} f_1 & f_2 & \cdots & f_m \\ f_1' & f_2' & \cdots & f_m' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \cdots & f_m^{(m-1)} \end{pmatrix}, \tag{4.15}$$

where $f_1, f_2, \cdots, f_m$ are functions of x and $f' := \partial f/\partial x, f'' := \partial^2 f/\partial x^2, f^{(m)} := \partial^m f/\partial x^m$ and so on.

In the commutative limit, $\Phi_N \star f$ is reduced to

$$\Phi_N \star f \longrightarrow \frac{\det W(f_1, f_2, \ldots, f_N, f)}{\det W(f_1, f_2, \ldots, f_N)}, \tag{4.16}$$

which just coincides with commutative one [25]. In this respect, quasi-determinants are fit to this framework of the Wronskian solutions.

From eq. (4.13), we have a more explicit form as

$$u_2 = \partial_x \left(\sum_{s=1}^N W_s' \star W_s^{-1} \right), \tag{4.17}$$

where

$$W_s := |W(f_1, \ldots, f_s)|_{ss}. \tag{4.18}$$

The l-reduction condition $(L^l)_{\leq -1} = 0$ or $L^l = B_l$ is realized at the level of the soliton solutions by taking $\alpha_s^l = \beta_s^l$ or equivalently $\alpha_s = \epsilon \beta_s$ for $s = 1, \cdots, N$, where ϵ is the l-th root of unity.

The present discussion is straightforwardly applicable for NC versions of the matrix KP hierarchy [32, 24, 25], the toroidal (matrix) GD hierarchy [33 – 38] and the (2-dimensional) Toda lattice hierarchy [39] formulated by pseudo-differential operators, because on commutative spaces, their exact soliton solutions are described by determinants of (generalized) Wronski matrices.

For example, we can give exact N-soliton solutions of the NC toroidal lKdV hierarchy $(l \geq 2)$[3] which is defined as follows.

First, we introduce two kind of infinite variables $\vec{x} = (x_1, x_2, \cdots)$ and $\vec{y} := (y_0, y_l, y_{2l}, \cdots)$ with $(x, y) \equiv (x_1, y_0)$. Noncommutativity is introduced into these coordinates. Next let us define two kind of Lax operators with respect to x, that is, an l-th order differential operator $P = (L)_{\geq 0}^l$ and a 0-th order pseudo-differential operator Q, where the coefficients depend on the two kind of infinite variables. An differential operator

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[3]Toroidal lKdV hierarchy is one of generalizations of lKdV hierarchy and first studied by Bogoyavlenskii [33] for $l = 2$ and developed by Billig, Iohara, Saito, Wakimoto, Ikeda and Takasaki where the symmetry of the solution space is revealed to be described in terms of a toroidal Lie algebra, that is, a central extension of double loop algebra $\mathcal{G}^{\mathrm{tor}} = \mathcal{SL}_l^{\mathrm{tor}}$ [35 – 37]. Hence we call it *toroidal lKdV hierarchy* or *toroidal GD hierarchy* in the present paper.

C_{ml} is also introduced in terms of P and Q as $C_{ml} := -(P^m \star Q)_{\geq 0}$. Then we can obtain the NC toroidal lKdV hierarchy:

$$\frac{\partial P}{\partial x_m} = [B_m, P]_\star, \qquad \qquad \frac{\partial Q}{\partial x_m} = [B_m, Q - \partial_y]_\star,$$

$$\frac{\partial P}{\partial y_{ml}} = [P^m \partial_y + C_{ml}, P]_\star, \qquad \frac{\partial Q}{\partial y_{ml}} = [P^m \partial_y + C_{ml}, Q - \partial_y]_\star. \qquad (4.19)$$

For $l = 2$, this includes the NC Calogero-Bogoyavlenskii-Schiff equation [40].

The N-soliton solution is given by

$$P = \Phi_N \star \partial_x^l \Phi_N^{-1}, \quad Q = (\partial_y \Phi_N) \star \Phi_N^{-1}, \qquad (4.20)$$

where the arguments in Φ_N is modified as follows:

$$f_s(\vec{x}, \vec{y}) := e_\star^{\xi_{rs}(\vec{x}, \vec{y}; \alpha_s)} + a_s e_\star^{\xi_{rs}(\vec{x}, \vec{y}; \beta_s)}, \qquad (4.21)$$

$$\xi_r(\vec{x}, \vec{y}; \alpha) := \xi(\vec{x}; \alpha) + r\xi(\vec{y}; \alpha)$$
$$= x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots + r y_0 + r y_l \alpha^l + r y_{2l} \alpha^{2l} + \cdots, \qquad (4.22)$$

with $\alpha_s^l = \beta_s^l$, where r is a constant. The proof is the same as the commutative one. (For the details, see section 5.1 in [37].) A key point of the proof is to show the evolution equations of Φ_N:

$$\frac{\partial \Phi_N}{\partial x_m} = -(\Phi_N \partial_x^m \Phi_N^{-1})_{\leq -1} \star \Phi_N = B_m \star \Phi_N - \Phi_N \partial_x^m,$$

$$\frac{\partial \Phi_N}{\partial y_{ml}} = (P^m \star Q)_{\leq -1} \star \Phi_N = (P^m \partial_y + C_{ml}) \star \Phi_N,$$

where the following property of quasideterminant plays crucial roles:

$$\Phi_N \star f_s = |W(f_1, \ldots, f_N, f_s)|_{N+1, N+1} = 0, \quad \text{for} \quad s = 1, \cdots, N.$$

This hierarchy generally gives rise to $(2+1)$-dimensional integrable equations where space and time coordinates are (x, y) and some other coordinate, respectively.

5. Asymptotic behavior of the exact soliton solutions

In this section, we discuss asymptotic behavior of the multi-soliton solutions at spatial infinity or infinitely past and future. In the star-product formalism, all coordinates are regarded as c-numbers and hence we can plot the configurations as usual and interpret the positions of localized wave packets, effect of coordinate shifts and so on as usual. Here we restrict ourselves to NC KdV and KP hierarchies, however, this observation would be also applicable to other NC hierarchies.

First, we present some special properties of the star exponential functions relevant to behavior of NC soliton solutions. In this section, we restrict ourselves to a specific equation on $(2+1)$ or $(1+1)$-dimensional space-time and noncommutativity should be introduce to some two specific space-time coordinates. Let us suppose that the specified NC coordinates are denoted by x_i and x_j $(i < j)$ which satisfies $[x_i, x_j]_\star = i\theta$.

First, let us comment on an important formula which is relevant to one-soliton solutions. Defining new coordinates $z := x_i + vx_j, \bar{z} := x_i - vx_j$, we can easily see

$$f(z) \star g(z) = f(z)g(z) \tag{5.1}$$

because the Moyal-product (2.2) is rewritten in terms of $(z, \bar{z})$ as [41]

$$f(z, \bar{z}) \star g(z, \bar{z}) = e^{iv\theta(\partial_{\bar{z}'}\partial_{z''} - \partial_{z'}\partial_{\bar{z}''})} f(z', \bar{z}')g(z'', \bar{z}'')\Big|_{\substack{z' = z'' = z \\ \bar{z}' = \bar{z}'' = \bar{z}.}} \tag{5.2}$$

Hence NC one soliton-solutions are essentially the same as commutative ones for both space-time and space-space noncommutativity cases.

When $f(x)$ is a linear function, the treatment of $e_\star^{f(x)}$ becomes easy as follows:

$$(e_\star^{\xi(\vec{x};\alpha)})^{-1} = e_\star^{-\xi(\vec{x};\alpha)}, \tag{5.3}$$

$$\partial_x e_\star^{\xi(\vec{x};\alpha)} = \alpha e_\star^{\xi(\vec{x};\alpha)}. \tag{5.4}$$

The proof can be seen as well from the fact that because of eq. (5.2), the star exponential function of a linear function itself reduces to commutative one, that is, $e_\star^{\xi(\vec{x},\alpha)} = e^{\xi(\vec{x},\alpha)}$. These formula are crucial in discussion on asymptotic behavior of N-soliton solutions.

Furthermore, the Baker-Campbell-Hausdorff (BCH) formula implies

$$e_\star^{\xi(\vec{x};\alpha)} \star e_\star^{\xi(\vec{x};\beta)} = e^{(i/2)\theta(\alpha^i\beta^j - \alpha^j\beta^i)} e_\star^{\xi(\vec{x};\alpha) + \xi(\vec{x};\beta)}. = e^{i\theta(\alpha^i\beta^j - \alpha^j\beta^i)} e_\star^{\xi(\vec{x};\beta)} \star e_\star^{\xi(\vec{x};\alpha)}. \tag{5.5}$$

The factor $(i/2)\theta(\alpha^i\beta^j - \alpha^j\beta^i)$ can be absorbed by a coordinate shift in $\xi(\vec{x};\alpha)$, and hence there is a possibility that noncommutativity might affect coordinate shifts by the factor such as phase shifts in the asymptotic behavior. When coordinates and fields are treated as complex, such a coordinate shift by a complex number causes no problem. However, if we want to apply NC integrable equations to real phenomena, such as, shallow water waves, then it becomes hard to interpret physically. Let us see what happens in the asymptotic region.

5.1 Asymptotic behavior of NC KdV hierarchy

First, let us discuss the NC KdV hierarchy and the asymptotic behavior of the N-soliton solutions. The NC KdV hierarchy is the 2-reduction of the NC KP hierarchy and realized by putting $\beta_s = -\alpha_s$ on the N-soliton solutions of the NC KP hierarchy. Here the constants α_s and a_s are non-zero real numbers and a_s is positive. Because of the permutation property of the columns of quasi-determinants (cf. section 1.1 in [18]), we can assume $\alpha_1 < \alpha_2 < \cdots < \alpha_N$.

In the NC KdV hierarchy, the x^{2n}-th flow becomes trivial and in the x^{2n+1}-th flow equation, space and time coordinates are specified as $(x, t) \equiv (x_1, x_{2n+1})$.

Now let us define a new coordinate $\tilde{x} := x + \alpha_I^{2n}t$ comoving with the I-th soliton and take $t \to \pm\infty$ limit.[4] Then, because of $x + \alpha_s^{2n}t = x + \alpha_I^{2n}t + (\alpha_s^{2n} - \alpha_I^{2n})t$, either $e_\star^{\alpha_s(x+\alpha_s^{2n}t)}$

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[4]Such kind of observation for soliton scatterings in NC integrable equations is first seen in [42]. (See also [2].)

or $e_\star^{-\alpha_s(x+\alpha_s^{2n}t)}$ goes to zero for $s \neq I$. Hence the behavior of f_s becomes at $t \to +\infty$:

$$f_s(\vec{x}) \longrightarrow \begin{cases} a_s e_\star^{-\alpha_s(x+\alpha_s^{2n}t)} & s < I \\ e_\star^{\alpha_I(x+\alpha_I^{2n}t)} + a_I e_\star^{-\alpha_I(x+\alpha_I^{2n}t)} & s = I \\ e_\star^{\alpha_s(x+\alpha_s^{2n}t)} & s > I, \end{cases} \tag{5.6}$$

and at $t \to -\infty$:

$$f_s(\vec{x}) \longrightarrow \begin{cases} e_\star^{\alpha_s(x+\alpha_s^{2n}t)} & s < I \\ e_\star^{\alpha_I(x+\alpha_I^{2n}t)} + a_I e_\star^{-\alpha_I(x+\alpha_I^{2n}t)} & s = I \\ a_s e_\star^{-\alpha_s(x+\alpha_s^{2n}t)} & s > I. \end{cases} \tag{5.7}$$

We note that the s-th ($s \neq I$) column is proportional to a single exponential function $e_\star^{\pm\alpha_s(x+\alpha_s^2 t)}$ due to eq. (5.4). Because of the multiplication property of columns of quasi-determinants (cf. section 1.2 in [18]), we can eliminate a common invertible factor from the s-th column in $|A|_{ij}$ where $s \neq j$. (Note that this exponential function is actually invertible as is shown in eq. (5.3).) Hence the N-soliton solution becomes the following simple form where only the I-th column is non-trivial, at $t \to +\infty$:

$$\Phi_N \star f \to \begin{vmatrix} 1 & \cdots & 1 & e_\star^{\xi(\vec{x};\alpha_I)} + a_I e_\star^{-\xi(\vec{x};\alpha_I)} & 1 & \cdots & 1 & f \\ -\alpha_1 & \cdots & -\alpha_{I-1} & \alpha_I(e_\star^{\xi(\vec{x};\alpha_I)} - a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & \alpha_{I+1} & \cdots & \alpha_N & f' \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-\alpha_1)^{N-1} & \cdots & (-\alpha_{I-1})^{N-1} & \alpha_I^{N-1}(e_\star^{\xi(\vec{x};\alpha_I)} + (-1)^{N-1}a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & \alpha_{I+1}^{N-1} & \cdots & \alpha_N^{N-1} & f^{(N-1)} \\ (-\alpha_1)^N & \cdots & (\alpha_{I-1})^N & \alpha_I^N(e_\star^{\xi(\vec{x};\alpha_I)} + (-1)^N a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & \alpha_{I+1}^N & \cdots & \alpha_N^N & \boxed{f^{(N)}} \end{vmatrix},$$

and at $t \to -\infty$:

$$\Phi_N \star f \to \begin{vmatrix} 1 & \cdots & 1 & e_\star^{\xi(\vec{x};\alpha_I)} + a_I e_\star^{-\xi(\vec{x};\alpha_I)} & 1 & \cdots & 1 & f \\ \alpha_1 & \cdots & \alpha_{I-1} & \alpha_I(e_\star^{\xi(\vec{x};\alpha_I)} - a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & -\alpha_{I+1} & \cdots & -\alpha_N & f' \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_1^{N-1} & \cdots & \alpha_{I-1}^{N-1} & \alpha_I^{N-1}(e_\star^{\xi(\vec{x};\alpha_I)} + (-1)^{N-1}a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & (-\alpha_{I+1})^{N-1} & \cdots & (-\alpha_N)^{N-1} & f^{(N-1)} \\ \alpha_1^N & \cdots & \alpha_{I-1}^N & \alpha_I^N(e_\star^{\xi(\vec{x};\alpha_I)} + (-1)^N a_I e_\star^{-\xi(\vec{x};\alpha_I)}) & (-\alpha_{I+1})^N & \cdots & (-\alpha_N)^N & \boxed{f^{(N)}} \end{vmatrix}.$$

Here we can see that all elements in between the first column and the N-th column commute and depend only on $x + \alpha_I^{2n}t$ in $\xi(\vec{x};\alpha_I)$, which implies that the corresponding asymptotic configuration coincides with the commutative one,[5] that is, the I-th one-soliton configuration with some coordinate shift so called the *phase shift*. The commutative discussion has been studied in this way by many authors, and therefore, we conclude that for the NC KdV hierarchy, *asymptotic behavior of the multi-soliton solutions is all the same as commutative one*, and as the results, *the N-soliton solutions possess N localized energy densities and in the scattering process, they never decay and preserve their shapes and velocities of the localized solitary waves. The phase shifts also occur by the same degree as commutative ones.*

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[5]Note that because f is arbitrary, there is no need to consider the products between a column and the $(N + 1)$-th column. This observation for asymptotic behavior can be made from eq. (4.17) also.

Finally, we make a brief comment on the 2-soliton solutions. In this situation, space-time dependence appears only as two kind of exponential factors $e^{\pm \alpha_1(x + \alpha_1^{2n}t)}$ and $e^{\pm \alpha_2(x + \alpha_2^{2n}t)}$. Noncommutativity of them could have effects by the factor $e^{\pm(i/2)\alpha_1\alpha_2(\alpha_1^{2n} - \alpha_2^{2n})\theta}$ because of the BCH formula. However, if the two kind of parameters satisfy

$$\frac{1}{2}\alpha_1\alpha_2(\alpha_1^{2n} - \alpha_2^{2n})\theta = 2\pi k, \tag{5.8}$$

where k is an non-zero integer, then, the effects of noncommutativity perfectly disappear at the every stage of calculations and the behavior of the 2-soliton solution perfectly coincides with that of commutative one at any time and any location. However the condition (5.8) is given specially by hand, and the mathematical and physical meaning of this observation is still unknown.

5.2 Asymptotic behavior of NC KP hierarchy

Now, let us discuss the NC KP hierarchy and the asymptotic behavior of the N-soliton solutions. The space and time coordinates are $(x, y, t) \equiv (x_1, x_2, x_n)$ and noncommutativity is introduced into some specified two coordinates among x, y and t. The specified NC coordinates are also denoted by x_i and x_j with $[x_i, x_j]_\star = i\theta$. Here the constants α_s and β_s are non-zero real numbers and the constant a_s will be redefined later.

As we mentioned at the beginning of the present section, one-soliton solutions are all the same as commutative ones. However, we have to treat carefully for the NC KP hierarchy. From eq. (4.17), naive one-soliton solution can be expressed as follows

$$u_2 = \partial_x \left(\partial_x (e_\star^{\xi(\vec{x};\alpha)} + a e_\star^{\xi(\vec{x};\beta)}) \star (e_\star^{\xi(\vec{x};\alpha)} + a e_\star^{\xi(\vec{x};\beta)})^{-1} \right)$$
$$= \partial_x \left((\alpha^i + a\beta^i \Delta e_\star^{\eta(\vec{x};\alpha,\beta)}) \star (1 + a\Delta e_\star^{\eta(\vec{x};\alpha,\beta)})^{-1} \right), \tag{5.9}$$

where

$$\eta(\vec{x}; \alpha, \beta) := x(\beta - \alpha) + y(\beta^2 - \alpha^2) + t(\beta^n - \alpha^n)$$
$$\Delta := e^{\frac{i}{2}\theta(\alpha^i\beta^j - \alpha^j\beta^i)}. \tag{5.10}$$

We note that the factor Δ can be absorbed by redefining a coordinate such as $x \to x + (\beta - \alpha)^{-1}(i/2)\theta(\alpha^i\beta^j - \alpha^j\beta^i)$. The final form of the solution depend only on $x_i(\beta_I^i - \alpha_I^i) + x_j(\beta_I^j - \alpha_I^j)$ for NC coordinates and the Moyal products disappear. Hence there becomes no dependence of complex numbers, and the one-soliton solution is the same as commutative one in this sense. However now we treat the coordinates as real and it would be better to redefine a positive real number $\tilde{a}$ which satisfies $a = \tilde{a}\Delta^{-1}$, so that $f_1 = e_\star^{\xi(\vec{x};\alpha)} + a e_\star^{\xi(\vec{x};\beta)} = \left(1 + \tilde{a}e_\star^{\eta(\vec{x};\alpha,\beta)}\right) \star e_\star^{\xi(\vec{x};\alpha)}$, in order to avoid such a coordinate shift by a complex number.

This point becomes important for the multi-soliton solutions. The constants a_s in the N-soliton solution of the NC KP hierarchy should be replaced with a positive real number $\tilde{a}_s$ which satisfies $a_s = \tilde{a}_s\Delta_s^{-1}$ where $\Delta_s := e^{(i/2)\theta(\alpha_s^i\beta_s^j - \alpha_s^j\beta_s^i)}$, because the N-soliton configuration reduces to a (I-th) one-soliton configuration when we set $\alpha_s = \beta_s = 0$ for all $s(\neq I)$.

Let us define new coordinates comoving with the I-th soliton as follows:

$$p := x + \alpha_I y + \alpha_I^{n-1} t, \quad q := x + \beta_I y + \beta_I^{n-1} t. \tag{5.11}$$

Then the function $\xi(x, y, t; \alpha_s)$ can be rewritten in terms of the new coordinates as $\xi(p, q, x_r; \alpha_s) = A(\alpha_s)p + B(\alpha_s)q + C(\alpha_s)x_r$ where x_r is a specified coordinate among x, y and t, and $A(\alpha_s), B(\alpha_s)$ and $C(\alpha_s)$ are real constants depending on α_I, β_I and α_s. For example, in the case of $x_r \equiv t$, we can get from eq. (5.11)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\beta_I - \alpha_I} \begin{pmatrix} \beta_I p - \alpha_I q + \alpha_I \beta_I (\beta_I^{n-2} - \alpha_I^{n-2})t \\ -p + q + (\alpha_I^{n-1} - \beta_I^{n-1})t \end{pmatrix}, \tag{5.12}$$

and find

$$\begin{aligned}
\xi &= x + \alpha_s y + \alpha_s^{n-1} t \\
&= \frac{\beta_I - \alpha_s}{\beta_I - \alpha_I} p + \frac{\alpha_s - \alpha_I}{\beta_I - \alpha_I} q + \left\{ \alpha_s^{n-1} + \frac{\alpha_I \beta_I (\beta_I^{n-2} - \alpha_I^{n-2}) + \alpha_s (\alpha_I^{n-1} - \beta_I^{n-1})}{\beta_I - \alpha_I} \right\} t.
\end{aligned}$$

Here we suppose that $C(\alpha_s) \neq C(\beta_s)$ which corresponds to pure soliton scatterings.[6]

Now let us take $x_r \to \pm\infty$ limit, then, for the same reason as in the NC KdV hierarchy, we can see that the asymptotic behavior of f_s becomes:

$$f_s(\vec{x}) \longrightarrow \begin{cases} A_s e_\star^{\xi(\vec{x};\gamma_s)} & s \neq I \\ e_\star^{\xi(\vec{x};\alpha_I)} + a_I e_\star^{\xi(\vec{x};\beta_I)} & s = I \end{cases} \tag{5.13}$$

where A_s is some real constant whose value is 1 or a_s, and γ_s is a real constant taking a value of α_s or β_s. As in the case of the NC KdV hierarchy, the s-th $(s \neq I)$ column is proportional to a single exponential function and we can eliminate this factor from the s-th column. Hence in the asymptotic region $x_r \to \pm\infty$, the N-soliton solution becomes the following simple form where only the I-th column is non-trivial:

$$\begin{aligned}
\Phi_N \star f &\to \begin{vmatrix}
1 & \cdots & 1 & e_\star^{\xi(\vec{x};\alpha_I)} + a_I e_\star^{\xi(\vec{x};\beta_I)} & 1 & \cdots & 1 & f \\
\gamma_1 & \cdots & \gamma_{I-1} & \alpha_I e_\star^{\xi(\vec{x};\alpha_I)} + a_I \beta_I e_\star^{\xi(\vec{x};\beta_I)} & \gamma_{I+1} & \cdots & \gamma_N & f' \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\gamma_1^{N-1} & \cdots & \gamma_{I-1}^{N-1} & \alpha_I^{N-1} e_\star^{\xi(\vec{x};\alpha_I)} + a_I \beta_I^{N-1} e_\star^{\xi(\vec{x};\beta_I)} & \gamma_{I+1}^{N-1} & \cdots & \gamma_N^{N-1} & f^{(N-1)} \\
\gamma_1^N & \cdots & \gamma_{I-1}^N & \alpha_I^N e_\star^{\xi(\vec{x};\alpha_I)} + a_I \beta_I^N e_\star^{\xi(\vec{x};\beta_I)} & \gamma_{I+1}^N & \cdots & \gamma_N^N & \boxed{f^{(N)}}
\end{vmatrix} \\
&= \begin{vmatrix}
1 & \cdots & 1 & 1 + \tilde{a}_I e_\star^{\eta(\vec{x};\alpha_I,\beta_I)} & 1 & \cdots & 1 & f \\
\gamma_1 & \cdots & \gamma_{I-1} & \alpha_I + \tilde{a}_I \beta_I e_\star^{\eta(\vec{x};\alpha_I,\beta_I)} & \gamma_{I+1} & \cdots & \gamma_N & f' \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\gamma_1^{N-1} & \cdots & \gamma_{I-1}^{N-1} & \alpha_I^{N-1} + \tilde{a}_I \beta_I^{N-1} e_\star^{\eta(\vec{x};\alpha_I,\beta_I)} & \gamma_{I+1}^{N+1} & \cdots & \gamma_N^{N-1} & f^{(N-1)} \\
\gamma_1^N & \cdots & \gamma_{I-1}^N & \alpha_I^N + \tilde{a}_I \beta_I^N e_\star^{\eta(\vec{x};\alpha_I,\beta_I)} & \gamma_{I+1}^N & \cdots & \gamma_N^N & \boxed{f^{(N)}}
\end{vmatrix}.
\end{aligned}$$

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[6]The condition $C(\alpha_s) = C(\beta_s)$ could lead to soliton resonances in commutative case.

Here we can see that all elements between the first column and the N-th column commute and depend only on $x_i(\beta_I^i - \alpha_I^i) + x_j(\beta_I^j - \alpha_I^j)$ for NC coordinates, which implies that the corresponding asymptotic configuration coincides with the commutative one. Hence, we can also conclude that for the NC KP hierarchy, *asymptotic behavior of the multi-soliton solutions is all the same as commutative one in the process of pure soliton scatterings*, and as the results, *the N-soliton solutions possess N localized energy densities and in the pure scattering process, they never decay and preserve their shapes and velocities of localized solitary waves.* Asymptotic behavior of two-soliton solution of NC KP equation studied by Paniak [15] actually coincides with our result for $n = 3, N = 2$.

Now we restricted ourselves to the NC KP hierarchy, however, this observation would be also true of other kind of NC hierarchies, such as, the (2-dimensional) NC Toda lattice hierarchy [43], the NC toroidal GD hierarchy presented in section 4 and the NC matrix KP hierarchy [44] because the soliton solutions could be represented by such kind of (generalized) Wronski matrices here, and the asymptotic analysis would be almost the same.

6. Conclusion and discussion

In this paper, we studied exact multi-soliton solutions of NC integrable hierarchies, including NC KP and toroidal KP hierarchies and the reductions, in terms of quasi-determinants. We found that the asymptotic behavior of them could be all the same as commutative ones in the process of (pure) soliton scatterings. This implies that the exact soliton solutions are actually solitons in the sense that the configuration has localized energy densities and never decay, and the phase shifts also appear by the same degree as in the commutative case.

It would be reasonable that there is no difference in asymptotic behavior of pure soliton scatterings on between commutative and NC spaces, because in asymptotic region, star-products reduce to ordinary commutative products and the effect of noncommutativity disappears. These results imply that we cannot detect effects of noncommutativity of space-time by observing such soliton dynamics. However, total behavior of them is unknown and it is worth studying further to find different aspects of the NC soliton dynamics from commutative ones.

Dynamics in soliton resonances is also interesting. From the present results of pure soliton scatterings, we could naturally expect that the configurations in soliton resonances would not be affected by noncommutativity in asymptotic region though we might need to make further modifications in the multi-soliton solutions such as $a_s = \tilde{a}_s \Delta_s^{-1}$ for the NC KP equation. Quantum treatments of the soliton scatterings is also interesting, such as properties of factorized S-matrix of NC sine-Gordon model [45]. Furthermore, the existence of multi-soliton solutions is important in integrable systems and the present observations might be a hint to reveal NC Hirota's bilinearization, theory of tau-functions and the structure of solution spaces.

Acknowledgments

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